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Properties of a Multivariate Goodness-of-Fit Test

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PROPERTIES OF A MULTIVARIATE GOODNESS-OF-FIT TEST

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ABSTRACT

We present the results of a Monte Carlo Study of the power of a new multivariate goodness-of-fit test.

INTRODUCTION

Foutz [1] developed a new test of goodness-of-fit for multivariate distributions. The test can also be used to fit univariate distributions; in an earlier paper [2] the authors compared the Foutz test with the Chi-square and Kolmogorov-Smirnov test. The results indicated that the Foutz test is more powerful in detecting certain characteristics than the other two tests. This paper deals with the performance of the test when fitting multivariate distributions. More specifically we investigate the power of the test when fitting bivariate and trivariate normal distributions for various choices of the mean vector and the covariance matrix. In the second section we present a brief description of the Foutz test; a discussion of the simulation procedure is in the third section and the results of the simulation are in the final section.

THE FOUTZ TEST

Let $\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{n-1}$ be a random sample of size n-1 from a p-variate continuous distribution. The first step of the Foutz test is to divide the sample space into n statistically equivalent blocks $\beta_1, \beta_2, \ldots, \beta_n$ and then determine a "continuous empirical distribution function (CEDF)" H_n. The test statistic for the hypothesis that the true c·d·f is H is

$$F_n = \sup_{\text{all events } \beta} \left| P_n(\beta) - P_H(\beta) \right| \qquad (1)$$

where $P_n(\beta)$ and $P_H(\beta)$ are the empirical and hypothesized probability measures of an event β, computed from H_n and H respectively. An equivalent computational formula for F_n is

$$F_n = \sum_{i=1}^{n} \max \left[0, \frac{1}{n} - D_i\right]$$

where $D_i = P[X \varepsilon \beta_i | H]$.

A general procedure for the construction of statistically equivalent blocks is the following. Select n - 1 "cutting functions" $h_k(\underline{X})$, k = 1, 2,...,n-1, such that $h_k(\underline{X})$ has a continuous distribution, and a permutation $k_1, k_2, \ldots, k_{n-1}$ of the integers 1,2,...,n-1. Order the samples $\underline{X}_i$ according to the value of $h_{k_1}(\underline{X}_i)$; let $\underline{X}(k_1)$ be the vector associated with the k_1th order statistic. Partition the sample space into two blocks B_1 and B_2 defined by

$$B_1 = \{\underline{X}: h_{k_1}(\underline{X}) \le h_{k_1}[\underline{X}(k_1)]\} \text{ and } B_2 = B_1^c .$$

At the second step, if $k_2 < k_1$, order the k_1 sample vectors in B_1 according to $h_{k_2}(X)$ and let $\underline{X}(k_2)$ be the k_2th order statistic. Partition B_1 into two sub-blocks B_{11} and B_{12}; at this stage the sample space is partitioned into three blocks B_{11}, B_{12} and $B_{20} = B_2$. If $k_2 > k_1$, order the $n - 1 - k_1$ sample values in the block B_2 according to $h_{k_2}(\underline{X})$. Let $\underline{X}(k_2)$ be the $(k_2 - k_1)$th order statistic and partition the block B_2 into two sub-blocks B_{21} and B_{22}; take $B_{10} = B_1$. Continue the process until all the cutting functions are exhausted; this results in a partition of the sample space into n statistically equivalent blocks $\beta_1, \beta_2, \ldots, \beta_n$. More details on the procedure and some examples are available in [3].

The null c·d·f of the test statistic F_n (necessary to determine the critical values) is quite difficult to derive even for small n; for n = 3, 4, 5 formulas for the exact c·d·f are in [2]. Foutz proposed a large sample normal approximation with mean $\mu = e^{-1}$ and $\sigma^2 = (2e^{-1} - 5e^{-2})/n$. In our earlier study [2] we found that with this approximation the observed significance level is about 10-20% smaller than the nominal values for sample size n - 1 = 20, 30, 50. We therefore proposed the use of empirically generated (based on 80,000 simulated F_n values) critical points in Table I below.

TABLE I. EMPIRICAL CRITICAL VALUES FOR FOUTZ TEST

Sample Size	20	30	50
Significance level			
.10	.42714	.41903	.40816
.05	.44865	.43533	.42116
.01	.48659	.46579	.44487

We have also generated a variation of the large sample normal approximation to calculate more accurate estimates of the critical values. We are presently in the process of using this approximation to generate tables for a spectrum of critical values and sample sizes.

SIMULATION PROCEDURE

To investigate the power of the Foutz test 5,000 replicates each of samples of size 20 from several bivariate and trivariate normal distributions were generated; in all cases the hypothesis tested was that the samples are from a bivariate/trivariate normal distribution with zero mean vector and covariance matrix the Identity. The true values of the means, variances and covariances for the generated samples were chosen so as to study the effect of (i) shifts in the means only (ii) shifts in the variances only (iii) shifts in covariances only and few cases involving a combination of all three.

The method of blocking we implemented was as follows. We let the samples themselves implicitly determine the permutation $k_1, k_2, \ldots, k_{n-1}$ and the order of the cutting functions which were all taken to be coordinate functions i.e., $h_{k_j}(\underline{X}) = \underline{X}^{(i)}$ the i^{th} coordinate of the sample vector $\underline{X}$, for some i. The following example with p = 2 (bivariate samples) will illustrate the procedure. Suppose $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_{n-1}$ are the observed sample vectors. The first cut on the p-dimensional sample space is made at $\underline{x}_1^{(1)}$ the first coordinate of the first sample vector $\underline{x}_1$. This partitions the sample space into two blocks B_1, B_2 defined by

	1^{st} coordinate	2^{nd} coordinate
B_1	$(-\infty, \underline{x}_1^{(1)}]$	$(-\infty, +\infty)$
B_2	$(\underline{x}_1^{(1)}, +\infty)$	$(-\infty, +\infty)$

The second sample vector $\underline{x}_2$ will be in one of the two blocks B_1 or B_2. Assume that it is in B_2; partition B_2 into two sub-blocks

	1^{st} coordinate	2^{nd} coordinate
B_{21}	$(\underline{x}_1^{(1)}, \infty)$	$(-\infty, \underline{x}_2^{(2)}]$
B_{22}	$(\underline{x}_1^{(1)}, +\infty)$	$(\underline{x}_2^{(2)}, +\infty)$

where $\underline{x}_2^{(2)}$ is the value of the second coordinate of the sample vector $\underline{x}_2$. At this stage the sample space is partitioned into three blocks B_{10}, B_{21} and B_{22}. Continuing this process (letting the cutting function at the q^{th} stage to be $\underline{x}_q^{(r)}$ the r^{th} coordinate of the $\underline{x}_q$ where $r = [\ (q-1) \bmod p + 1]$) until all the sample vectors are exhausted will result in a partition of the sample space into n statistically equivalent blocks. In our simulation we used both rectangular and polar/spherical coordinates to examine if one scheme is more adept at detecting certain types of violations of the null hypothesis. When using spherical coordinates the first coordinate is taken to be the radius vector (same for polar coordinates), the second coordinate is the angle in the horizontal plane and the third coordinate is the angle measured from the vertical axis. Figures 1 and 2 graphically demonstrate the construction of the blocks for five bivariate samples using the rectangular and polar coordinate systems. The implicit permutation and the order of the coordinate cutting functions are also included in the figures. It should be noted that for polar/spherical coordinates it is necessary to make an additional initial cut, which we took at $\theta = 0°$.

The probability contents of the statistically equivalent blocks under the null hypothesis (multivariate normal with Zero mean vector and covariance matrix the Identity) are easily computable as products of univariate probabilities (normal, Chi square and uniform) for the rectangular coordinate system as well as the polar/spherical coordinate system.

RESULTS

For ease of comprehension, the results for shifts of mean or variance/covariance are given as power curves in Figures 3 - 7. All power curves are based on 5,000 replications at the .05 significance level. The results are indicative of those obtained for significance levels of .01 and .1. Full details are available in

Linhart [3].

Shifts in mean are detected well. Figures 3 and 4 show that a shift of one standard deviation in the mean results in about a 60% rejection rate for both bivariate and trivariate samples. The rectangular method of blocking consistently resulted in a higher rejection rate than did the polar/spherical method.

Power curves for shifts in variance are given in Figures 5 and 6. Small shifts in variance are not detected very well, but larger shifts and shifts in more than one component resulted in higher rejection rates. Neither blocking method produced rejection rates significantly better than the other except in the trivariate case when one variance was shifted. In general, the polar/spherical method of blocking gives slightly higher rejection rates, but not consistently.

The power curves for shifts in covariance are given in Figure 7. Except for highly correlated data, neither blocking scheme detects these shifts very well. The polar/spherical method of blocking usually gives somewhat higher rejection rates than the rectangular method.

To test for possible confounding in detection of shifts in both mean and variance/covariance, we made the series of runs listed in Tables II and III. Shifts are generally of increasing amount as one looks down and to the right in the tables. It is also true that rejection rates increase as one looks down and to the right, indicating there is no detectable confounding. The rectangular method of blocking generally gave better results with multiple shifts.

All of the above results are based on a sample size of 20 and significance level of .05. To determine possible trends over sample size or significance level we made the series of runs given in Tables IV and V. Larger sample sizes generally led to higher rejection rates, as is to be anticipated. Violations of this did occur when rejection rates were close to the significance level, i.e., the shift was not detected very well. The rectangular blocking method generally gave higher rejection rates.

In conclusion, the Foutz test is adept at detecting shifts in mean, less powerful at detecting shifts in variance, and poor in detecting correlated variates unless they are highly correlated. In addition, shifts in mean are not disguised when a shift in variance/covariance is also present.

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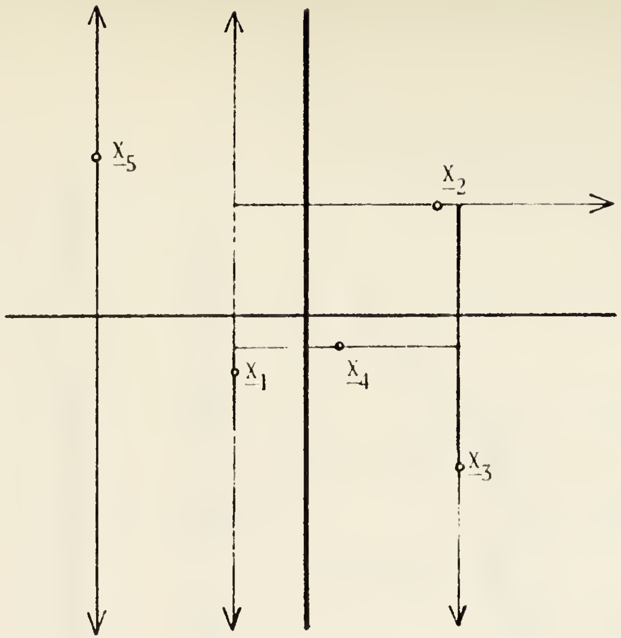
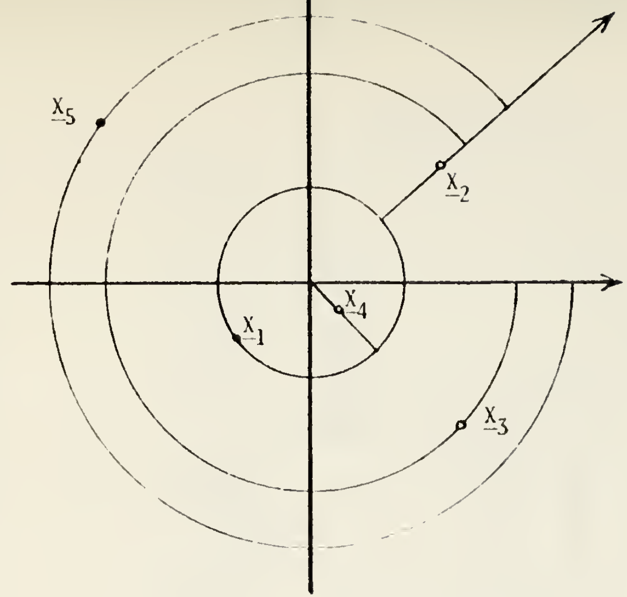


i	$h_i(\underline{X})$	k_i
1	$X^{(1)}$	2
2	$X^{(1)}$	5
3	$X^{(2)}$	4
4	$X^{(1)}$	3
5	$X^{(2)}$	1

FIG. 1: RECTANGULAR COORDINATE BLOCKING



i	$h_i(\underline{X})$	k_i
1	$X^{(2)}$	2
2	$X^{(1)}$	3
3	$X^{(2)}$	4
4	$X^{(1)}$	1
5	$X^{(1)}$	5

FIG. 2: POLAR COORDINATE BLOCKING

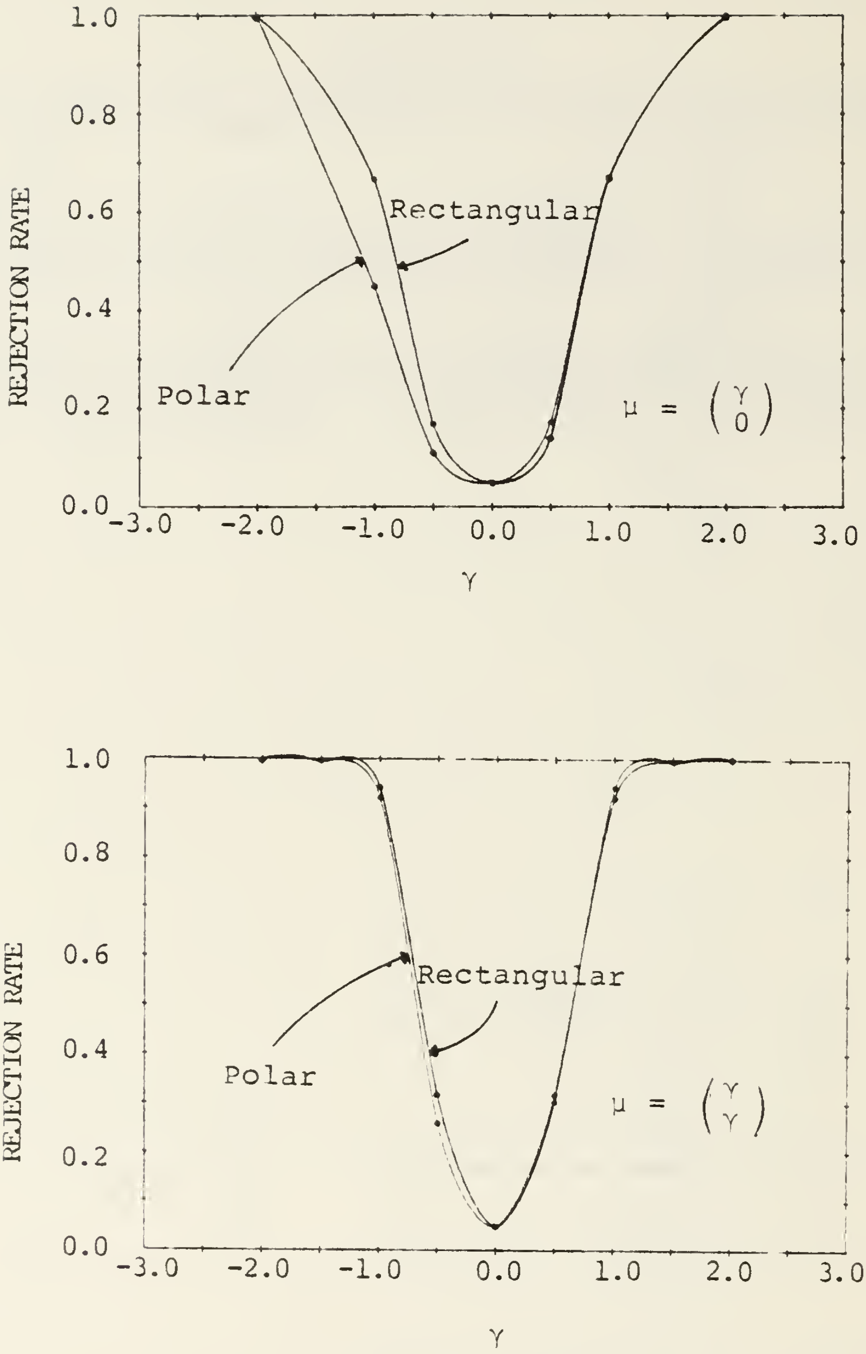
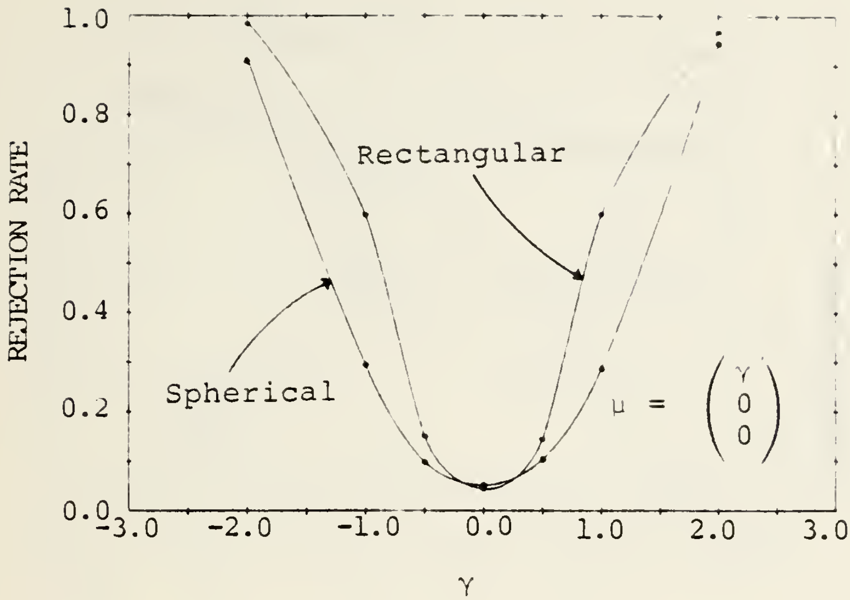


FIGURE 3: POWER CURVES FOR SHIFTS IN MEAN



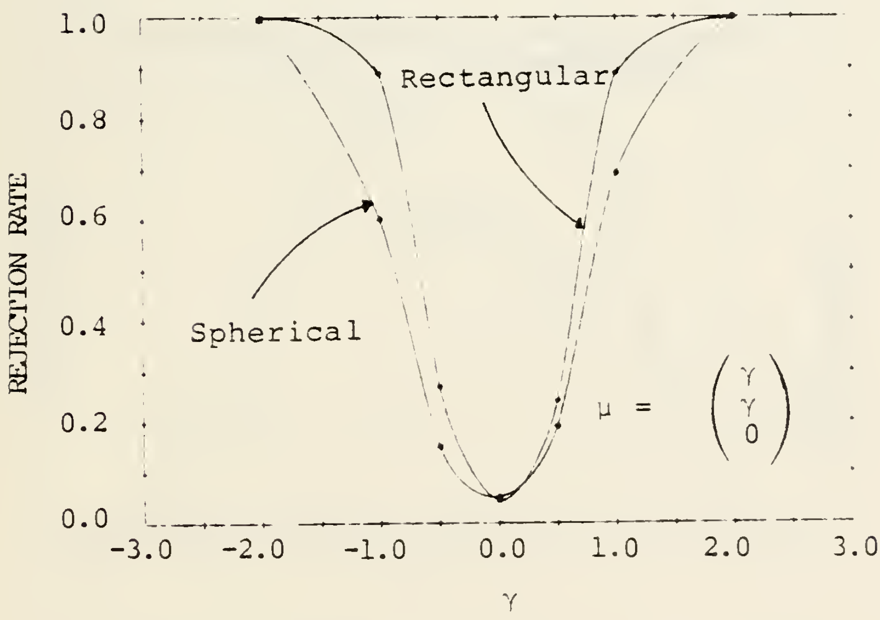


FIGURE 4: POWER CURVES FOR SHIFTS IN MEAN

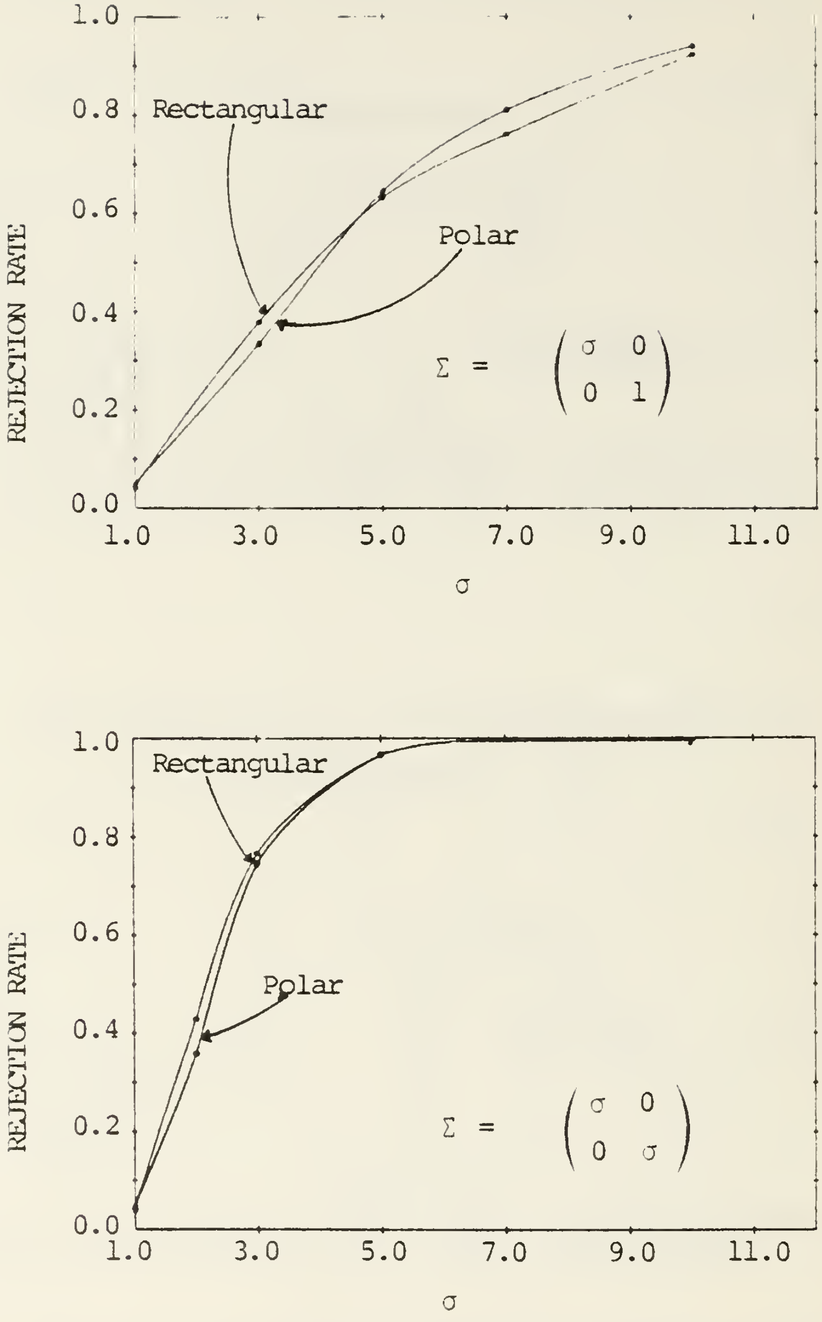


FIGURE 5: POWER CURVES FOR SHIFTS IN VARIANCE

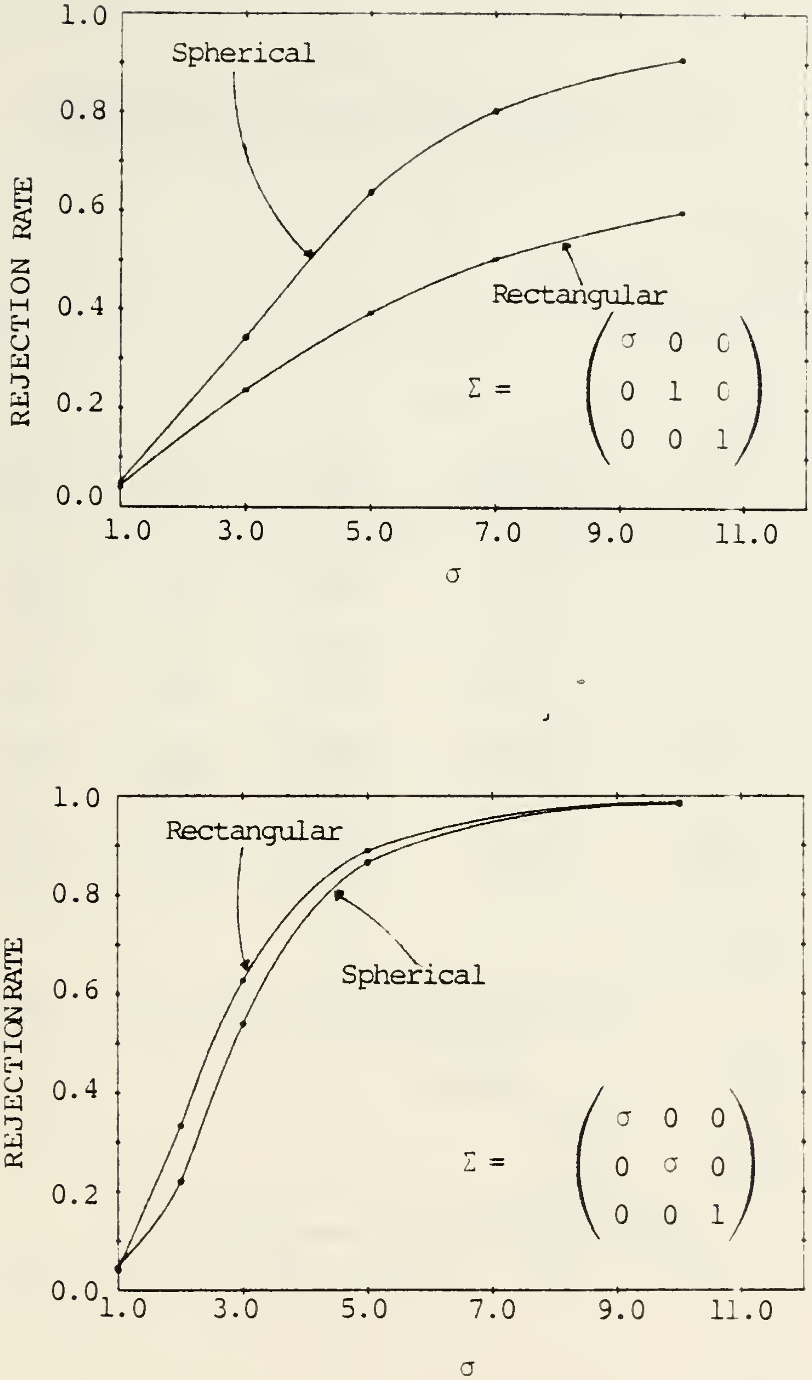


FIGURE 6: POWER CURVES FOR SHIFTS IN VARIANCE

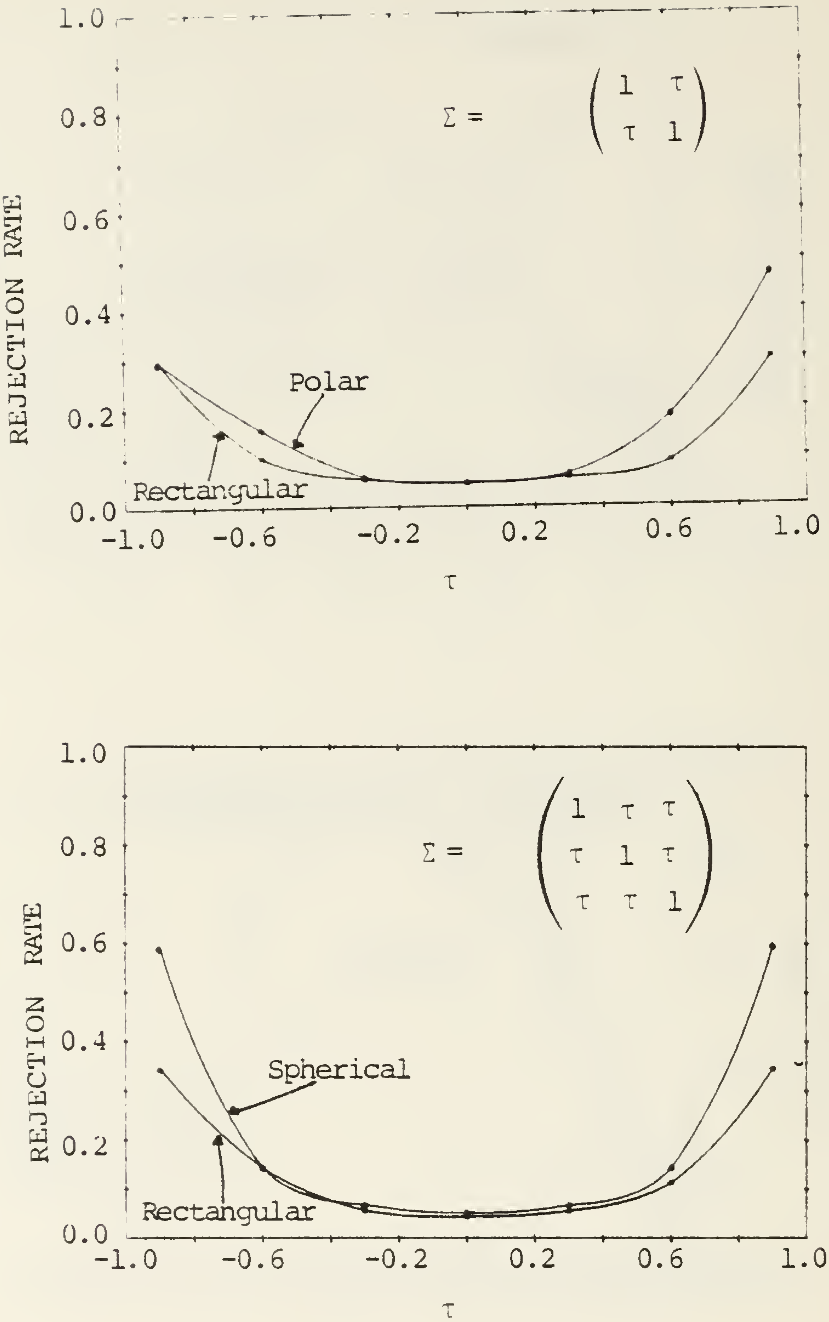


FIGURE 7: POWER CURVES FOR SHIFTS IN COVARIANCE

TABLE II

REJECTION RATES FOR MULTIPLE SHIFTS IN
MEAN AND VARIANCE-COVARIANCE

$\alpha = .05,\ n = 21$*

Sigma	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & .6\\.6 & 1\end{pmatrix}$	$\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$	$\begin{pmatrix}2 & .849\\.849 & 2\end{pmatrix}$	$\begin{pmatrix}5 & 1.34\\1.34 & 5\end{pmatrix}$
Mean					
$\begin{pmatrix}0\\0\end{pmatrix}$	.0488	.0912	.1986	.2500	.9658
	.0482	.1176	.1522	.2162	.9572
$\begin{pmatrix}.5\\0\end{pmatrix}$	.1710	.2398	.3110	.3702	.9720
	.1388	.2482	.2498	.3402	.9650
$\begin{pmatrix}1\\0\end{pmatrix}$	.5606	.7346	.6384	.6828	.9820
	.4348	.5952	.5334	.6316	.9764
$\begin{pmatrix}1\\1\end{pmatrix}$	.9418	.8774	.9350	.8658	.9892
	.8576	.8588	.8722	.8308	.9840
$\begin{pmatrix}2\\0\end{pmatrix}$	.9998	.9998	.9902	.9950	.9990
	.9950	.9990	.9772	.9882	.9964

* First Entry - Rectangular coordinate blocking
Second Entry - Polar coordinate blocking

TABLE III

REJECTION RATES FOR MULTIPLE SHIFTS IN MEAN AND VARIANCE-COVARIANCE

$\alpha = .05,\ n = 21$*

Mean \ Sigma	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & .6 \\ 0 & 1 & 0 \\ .6 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 10 & 0 & .95 \\ 0 & 1 & 0 \\ .95 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$
$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	.0440 .0518	.0674 .0582	.5392 .4584	.7828 .7840	.9770 .9720
$\begin{pmatrix} .5 \\ 0 \\ 0 \end{pmatrix}$	.0480 .0280	.1830 .1176	.5708 .5034	.7946 .8020	.9832 .9740
$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	.3728 .1174	.6352 .2912	.6852 .6254	.8206 .8422	.9888 .9824
$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$	.7400 .7392	.9074 .7454	.9270 .8602	.9668 .9454	.9930 .9870
$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$	.9982 .9726	.9956 .9752	.9716 .9742	.9736 .9774	.9978 .9976

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* First Entry - Rectangular coordinate blocking
Second Entry - Polar coordinate blocking

TABLE IV

REJECTION RATES FOR INCREASING SAMPLE SIZES

Sample size (n-1)	20	30	50
Shift			
		$\alpha = .01$	
$\mu = \begin{pmatrix} .5 \\ 0 \end{pmatrix}$	.0574 .0430	.0860 .0564	.1270 .0754
$\mu = \begin{pmatrix} .5 \\ .5 \end{pmatrix}$	.1294 .1230	.2026 .1418	.3652 .2508
$\Sigma = \begin{pmatrix} 1 & .3 \\ .3 & 1 \end{pmatrix}$	.0126 .0136	.0140 .0152	.0176 .0170
$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$	.1864 .1578	.2722 .2244	.4522 .3744
		$\alpha = .05$	
$\mu = \begin{pmatrix} .5 \\ 0 \end{pmatrix}$	.1710 .1388	.2170 .1630	.2914 .2238
$\mu = \begin{pmatrix} .5 \\ .5 \end{pmatrix}$	.3024 .3164	.4144 .3076	.6030 .4826
$\Sigma = \begin{pmatrix} 1 & .3 \\ .3 & 1 \end{pmatrix}$	.0576 .0656	.0624 .0624	.0728 .0760
$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$	.3786 .3342	.4884 .4304	.6756 .6016
		$\alpha = .10$	
$\mu = \begin{pmatrix} .5 \\ 0 \end{pmatrix}$	.2700 .2298	.3228 .2658	.4256 .3400
$\mu = \begin{pmatrix} .5 \\ .5 \end{pmatrix}$	.4406 .4534	.5424 .4336	.7190 .6066
$\Sigma = \begin{pmatrix} 1 & .3 \\ .3 & 1 \end{pmatrix}$	.1178 .1258	.1174 .1196	.1396 .1422
$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$	.5150 .4640	.6132 .5566	.7800 .7160

First Entry - Rectangular coordinate blocking
Second Entry - Polar coordinate blocking

TABLE V

REJECTION RATES FOR INCREASING SAMPLE SIZES

Sample size (n-1)	20	30	50
Shift		α = .01	
$\mu = \begin{pmatrix} .5 \\ 0 \\ 0 \end{pmatrix}$	.0480 .0280	.0680 .0362	.1036 .0526
$\mu = \begin{pmatrix} .5 \\ .5 \\ .5 \end{pmatrix}$	.1738 .0848	.2932 .1662	.5040 .3428
$\Sigma = \begin{pmatrix} 1 & 0 & .3 \\ 0 & 1 & 0 \\ .3 & 0 & 1 \end{pmatrix}$	.0106 .0124	.0138 .0134	.0148 .0144
$\Sigma = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	.1606 .0838	.2054 .1138	.3528 .2080
		α = .05	
$\mu = \begin{pmatrix} .5 \\ 0 \\ 0 \end{pmatrix}$	.1438 .1036	.1974 .1256	.2742 .1646
$\mu = \begin{pmatrix} .5 \\ .5 \\ .5 \end{pmatrix}$	.3642 .2198	.5118 .3588	.7268 .5868
$\Sigma = \begin{pmatrix} 1 & 0 & .3 \\ 0 & 1 & 0 \\ .3 & 0 & 1 \end{pmatrix}$	.0468 .0512	.0588 .0488	.0656 .0540
$\Sigma = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	.3438 .2074	.3976 .2708	.5734 .4126
		α = .10	
$\mu = \begin{pmatrix} .5 \\ 0 \\ 0 \end{pmatrix}$	.2484 .1874	.3024 .2132	.3876 .2650
$\mu = \begin{pmatrix} .5 \\ .5 \\ .5 \end{pmatrix}$	.4948 .3584	.6396 .4912	.8272 .7040
$\Sigma = \begin{pmatrix} 1 & 0 & .3 \\ 0 & 1 & 0 \\ .3 & 0 & 1 \end{pmatrix}$	.0972 .1086	.1142 .1030	.1232 .1102
$\Sigma = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	.4736 .3156	.5264 .3880	.6880 .5450

First Entry - Rectangular coordinate blocking
Second Entry - Polar coordinate blocking

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